CANONICAL DECOMPOSITION OF A TETRABLOCK CONTRACTION AND OPERATOR MODEL

SOURAV PAL

Abstract. A triple of commuting operators for which the closed tetrablock $\overline{\mathbb{E}}$ is a spectral set is called a tetrablock contraction or an $\mathbb{E}$-contraction. The set $\mathbb{E}$ is defined as

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \,:\, 1 - zx_1 - wx_2 + zwx_3 \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\}.$$

We show that every $\mathbb{E}$-contraction can be uniquely written as a direct sum of an $\mathbb{E}$-unitary and a completely non-unitary $\mathbb{E}$-contraction. It is analogous to the canonical decomposition of a contraction operator into a unitary and a completely non-unitary contraction. We produce a concrete operator model for such a triple satisfying some conditions.

1. Introduction

A compact subset X of $\mathbb{C}^n$ is said to be a *spectral set* for a commuting n-tuple of bounded operators $\underline{T} = (T_1, \dots, T_n)$ defined on a Hilbert space $\mathcal{H}$ if the Taylor joint spectrum $\sigma(\underline{T})$ of $\underline{T}$ is a subset of X and

$$\|r(\underline{T})\| \leq \|r\|_{\infty, X} = \sup\{|r(z_1, \dots, z_n)| \,:\, (z_1, \dots, z_n) \in X\},$$

for all rational functions r in $\mathcal{R}(X)$. Here $\mathcal{R}(X)$ denotes the algebra of all rational functions on X, that is, all quotients p/q of holomorphic polynomials p, q in n-variables for which q has no zeros in X. A triple of commuting operators (A, B, P) for which the closure of the tetrablock $\mathbb{E}$, where

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \,:\, 1 - zx_1 - wx_2 + zwx_3 \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\},$$

is a spectral set is called a *tetrablock contraction* or an $\mathbb{E}$-*contraction*.

Complex geometry, function theory and operator theory on the tetrablock have been widely studied by a number of mathematicians [1, 2, 4, 5, 6, 7, 9, 11, 13] over past one decade because of the relevance of this domain to μ-synthesis problem and H^∞ control theory. The following result from [1] (Theorem 2.4 in [1]) characterizes points in $\mathbb{E}$ and $\overline{\mathbb{E}}$ and provides a geometric description of the tetrablock.

2010 *Mathematics Subject Classification.* 47A13, 47A15, 47A20, 47A25, 47A45.

Key words and phrases. Tetrablock contraction, Canonical Decomposition, Fundamental operators, Operator model.

The author was supported by the INSPIRE Faculty Award (Award No. DST/INSPIRE/04/2014/001462) of DST, India.

Theorem 1.1. *A point $(x_1, x_2, x_3) \in \mathbb{C}^3$ is in $\overline{\mathbb{E}}$ if and only if $|x_3| \leq 1$ and there exist $c_1, c_2 \in \mathbb{C}$ such that $|c_1| + |c_2| \leq 1$ and $x_1 = c_1 + \bar{c_2}x_3, \quad x_2 = c_2 + \bar{c_1}x_3$.*

It is clear from the above result that the closed tetrablock $\overline{\mathbb{E}}$ lives inside the closed tridisc $\overline{\mathbb{D}^3}$ and consequently an $\mathbb{E}$-contraction consists of commuting contractions. It is evident from the definition that if (A, B, P) is an $\mathbb{E}$-contraction then so is its adjoint (A^*, B^*, P^*). We briefly recall from literature some special classes of $\mathbb{E}$-contractions which are analogous to unitaries, isometries, co-isometries etc. in one variable operator theory.

Definition 1.2. Let A, B, P be commuting operators on a Hilbert space $\mathcal{H}$. We say that (A, B, P) is

(i) an $\mathbb{E}$-*unitary* if A, B, P are normal operators and the joint spectrum $\sigma(A, B, P)$ is contained in the distinguished boundary $b\overline{\mathbb{E}}$ of the tetrablock , where

$$\begin{aligned} b\overline{\mathbb{E}} &= \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 = \bar{x_2}x_3, |x_2| \leq 1, |x_3| = 1\} \\ &= \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : |x_3| = 1\}. \end{aligned}$$

(ii) an $\mathbb{E}$-*isometry* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an $\mathbb{E}$-unitary $(\tilde{A}, \tilde{B}, \tilde{P})$ on $\mathcal{K}$ such that $\mathcal{H}$ is a common invariant subspace of A, B, P and that $A = \tilde{A}|_{\mathcal{H}}, B = \tilde{B}|_{\mathcal{H}}, P = \tilde{P}|_{\mathcal{H}}$;

(iii) an $\mathbb{E}$-*co-isometry* if (A^*, B^*, P^*) is an $\mathbb{E}$-isometry ;

(iv) a *completely non-unitary* $\mathbb{E}$-contraction if $(A, B.P)$ is an $\mathbb{E}$-contraction and P is a completely non-unitary contraction ;

(v) a *pure* $\mathbb{E}$-contraction if $(A, B.P)$ is an $\mathbb{E}$-contraction and P is a pure contraction, that is, $P^{*n} \to 0$ strongly as $n \to \infty$.

Definition 1.3. Let (A, B, P) be an $\mathbb{E}$-contraction on a Hilbert space $\mathcal{H}$. A commuting triple (V_1, V_2, V_3) on $\mathcal{K}$ is said to be an $\mathbb{E}$-isometric dilation of (A, B, P) if (V_1, V_2, V_3) is an $\mathbb{E}$-isometry, $\mathcal{H} \subseteq \mathcal{K}$ and

$$f(A, B, P) = P_{\mathcal{H}} f(V_1, V_2, V_3)|_{\mathcal{H}}$$

for every holomorphic polynomial f in three variables. Here $P_{\mathcal{H}}$ denotes the projection onto $\mathcal{H}$. Moreover, this dilation is called minimal if

$$\mathcal{K} = \overline{\text{span}}\{f(V_1, V_2, V_3)h \,:\, h \in \mathcal{H}\,, f \in \mathbb{C}[z_1, z_2, z_3]\}.$$

It was a path breaking discovery by von Neumann, [12], that a bounded operator T is a contraction if and only if the closed unit disc $\overline{\mathbb{D}}$ in the complex plane is a spectral set for T. It is well known that to every contraction T on a Hilbert space $\mathcal{H}$ there corresponds a decomposition of $\mathcal{H}$ into an orthogonal sum of two subspaces reducing T, say $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $T|_{\mathcal{H}_1}$ is unitary and $T|_{\mathcal{H}_2}$ is completely non-unitary; $\mathcal{H}_1$ or $\mathcal{H}_2$ may equal the trivial subspace $\{0\}$. This decomposition is uniquely determined and is called the canonical decomposition of a contraction (see Theorem 3.2 in Ch-I, [10] for

details). Indeed, $\mathcal{H}_1$ consists of those elements $h \in \mathcal{H}$ for which

$$\|T^n h\| = \|h\| = \|T^{*n}h\| \qquad (n = 1, 2, \dots)\,.$$

The main aim of this article is to show that an $\mathbb{E}$-contraction admits an analogous decomposition into an $\mathbb{E}$-unitary and a completely non-unitary $\mathbb{E}$-contraction. Indeed, in Theorem 3.1, one of the main results of this paper, we show that for an $\mathbb{E}$-contraction (A, B, P) defined on $\mathcal{H}$ if $\mathcal{H}_1 \oplus \mathcal{H}_2$ is the unique orthogonal decomposition of $\mathcal{H}$ into reducing subspaces of P such that $P|_{\mathcal{H}_1}$ is a unitary and $P|_{\mathcal{H}_2}$ is a completely non-unitary, then $\mathcal{H}_1, \mathcal{H}_2$ also reduce A, B ; $(A|_{\mathcal{H}_1}, B|_{\mathcal{H}_1}, P|_{\mathcal{H}_1})$ is an $\mathbb{E}$-unitary and $(A|_{\mathcal{H}_2}, B|_{\mathcal{H}_2}, P|_{\mathcal{H}_2})$ is a completely non-unitary $\mathbb{E}$-contraction.

The other contribution of this article is that we produce a concrete operator model for an $\mathbb{E}$-contraction which satisfies some conditions. Before getting into the details of it we recall a few words from the literature about the fundamental equations and the fundamental operators related to an $\mathbb{E}$-contraction.

For an $\mathbb{E}$-contraction (A, B, P), the *fundamental equations* were defined in [4] as

(1.1)

$$A - B^*P = D_P X_1 D_P\,, \quad B - A^*P = D_P X_2 D_P\,; \qquad D_P = (I - P^*P)^{\frac{1}{2}}.$$

It was proved in [4] (Theorem 3.5, [4]) that corresponding to every $\mathbb{E}$-contraction (A, B, P) there were two unique operators F_1, F_2 in $\mathcal{B}(\mathcal{D}_P)$ that satisfied the fundamental equations, i.e,

$$A - B^*P = D_P F_1 D_P\,,\ B - A^*P = D_P F_2 D_P\,.$$

Here $\mathcal{D}_P = \overline{Ran}\, D_P$ and is called the defect space of P. Also $\mathcal{B}(\mathcal{H})$, for a Hilbert space $\mathcal{H}$, always denotes the algebra of bounded operators on $\mathcal{H}$. An explicit $\mathbb{E}$-isometric dilation was constructed for a particular class of $\mathbb{E}$-contractions in [4] (Theorem 6.1, [4]) and F_1, F_2 played the fundamental role in that explicit construction of dilation. For their pivotal role in the dilation, F_1 and F_2 were called the *fundamental operators* of (A, B, P).

It was shown in [4] (Theorem 6.1, [4]) that an $\mathbb{E}$-contraction (A, B, P) dilated to an $\mathbb{E}$-isometry if the corresponding fundamental operators F_1, F_2 satisfied $[F_1, F_2] = 0$ and $[F_1^*, F_1] = [F_2^*, F_2]$. Here $[S_1, S_2] = S_1S_2 - S_2S_1$ for any two bounded operators S_1, S_2. On the other hand there are $\mathbb{E}$-contractions which do not dilate. Indeed, an $\mathbb{E}$-contraction may not dilate to an $\mathbb{E}$-isometry if $[F_1^*, F_1] \neq [F_2^*, F_2]$; it has been established in [8] by a counter example. So it turns out that those two conditions are very crucial for an $\mathbb{E}$-contraction. In Theorem 4.4, we construct a concrete model for an $\mathbb{E}$-contraction (A, B, P) when the fundamental operators F_{1*}, F_{2*} of (A^*, B^*, P^*) satisfy $[F_{1*}, F_{2*}] = 0$ and $[F_{1*}^*, F_{1*}] = [F_{2*}^*, F_{2*}]$. In brief, such an $\mathbb{E}$-contraction is the restriction to a common invariant subspace of an $\mathbb{E}$-co-isometry and every $\mathbb{E}$-co-isometry is expressible as the orthogonal direct

sum of an $\mathbb{E}$-unitary and a pure $\mathbb{E}$-co-isometry, which has a model on the vectorial Hardy space $H^2(\mathcal{D}_{T_3})$, where T_3^* is the minimal isometric dilation of P^*.

In section 2, we accumulate a few new results about $\mathbb{E}$-contractions and also state some results from the literature which will be used in sequel.

2. The set $\mathbb{E}$ and $\mathbb{E}$-contractions

We begin this section with a lemma that characterizes the points in $\overline{\mathbb{E}}$.

Lemma 2.1. $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$ *if and only if* $(\omega x_1, \omega x_2, \omega^2 x_3) \in \overline{\mathbb{E}}$ *for all* $\omega \in \mathbb{T}$.

Proof. Let $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$. Then by Theorem 1.1, $|x_3| \leq 1$ and there are complex numbers c_1, c_2 with $|c_1| + |c_2| \leq 1$ such that $x_1 = c_1 + \bar{c}_2 x_3, \quad x_2 = c_2 + \bar{c}_1 x_3$. For $\omega \in \mathbb{T}$ if we choose $d_1 = \omega c_1$ and $d_2 = \omega c_2$ we see that $|d_1| + |d_2| \leq 1$ and

$$\begin{aligned} \omega x_1 &= \omega(c_1 + \bar{c}_2 x_3) = \omega c_1 + \overline{\omega c_2}(\omega^2 x_3) = d_1 + \bar{d}_2(\omega^2 x_3)\,, \\ \omega x_2 &= \omega(c_2 + \bar{c}_1 x_3) = \omega c_2 + \overline{\omega c_1}(\omega^2 x_3) = d_2 + \bar{d}_1(\omega^2 x_3). \end{aligned}$$

Therefore, by Theorem 1.1, $(\omega x_1, \omega x_2, \omega^2 x_3) \in \mathbb{E}$. The other side of the proof is trivial.

■

The following lemma simplifies the definition of $\mathbb{E}$-contraction.

Lemma 2.2. *A triple of commuting operators* (A, B, P) *is an* $\mathbb{E}$*-contraction if and only if*

$$\|f(A, B, P)\| \leq \|f\|_{\infty, \overline{\mathbb{E}}} = \sup\{|f(x_1, x_2, x_3)| \, : \, (x_1, x_2, x_3) \in \overline{\mathbb{E}}\}$$

for all holomorphic polynomials f *in three variables.*

This actually follows from the fact that $\overline{\mathbb{E}}$ is polynomially convex. A proof to this could be found in [4] (Lemma 3.3, [4]).

Lemma 2.3. *Let* (A, B, P) *be an* $\mathbb{E}$*-contraction. Then so is* $(\omega A, \omega B, \omega^2 P)$ *for any* $\omega \in \mathbb{T}$.

Proof. Let $f(x_1, x_2, x_3)$ be a holomorphic polynomial in the co-ordinates of $\overline{\mathbb{E}}$ and for $\omega \in \mathbb{T}$ let $f_1(x_1, x_2, x_3) = f(\omega x_1, \omega x_2, \omega^2 x_3)$. It is evident from Lemma 2.1 that

$$\sup\{|f(x_1, x_2, x_3)| \, : \, (x_1, x_2, x_3) \in \overline{\mathbb{E}}\} = \sup\{|f_1(x_1, x_2, x_3)| \, : \, (x_1, x_2, x_3) \in \overline{\mathbb{E}}\}.$$

Therefore,

$$\begin{aligned} \|f(\omega A, \omega B, \omega^2 P)\| &= \|f_1(A, B, P)\| \\ &\leq \|f_1\|_{\infty, \overline{\mathbb{E}}} \\ &= \|f\|_{\infty, \overline{\mathbb{E}}}. \end{aligned}$$

Therefore, by Lemma 2.2, $(\omega A, \omega B, \omega^2 P)$ is an $\mathbb{E}$-contraction. ■

The following result was proved in [4] (see Theorem 3.5 in [4]).

Theorem 2.4. *Let (A, B, P) be an $\mathbb{E}$-contraction. Then the operator functions ρ_1 and ρ_2 defined by*

$$\rho_1(A, B, P) = (I - P^*P) + (A^*A - B^*B) - 2 \; Re \; (A - B^*P) \, ,$$
$$\rho_2(A, B, P) = (I - P^*P) + (B^*B - A^*A) - 2 \; Re \; (B - A^*P)$$

satisfy

$$\rho_1(A, zB, zP) \geq 0 \text{ and } \rho_2(A, zB, zP) \geq 0 \text{ for all } z \in \overline{\mathbb{D}}.$$

Lemma 2.5. *Let (A, B, P) be an $\mathbb{E}$-contraction. Then for $i = 1, 2$, $\rho_i(\omega A, \omega B, \omega^2 P) \geq 0$ for all $\omega \in \mathbb{T}$.*

Proof. By Theorem 2.4,

$$\rho_1(A, B, P) \geq 0 \text{ and } \rho_2(A, B, P) \geq 0.$$

Since $(\omega A, \omega B, \omega^2 P)$ is an $\mathbb{E}$-contraction for every ω in $\mathbb{T}$ by Lemma 2.3, we have that

$$\rho_1(\omega A, \omega B, \omega^2 P) \geq 0 \text{ and } \rho_2(\omega A, \omega B, \omega^2 P) \geq 0 \, .$$

∎

The following theorem provides a set of characterizations for $\mathbb{E}$-unitaries and for a proof to this one can see Theorem 5.4 in [4].

Theorem 2.6. *Let $\underline{N} = (N_1, N_2, N_3)$ be a commuting triple of bounded operators. Then the following are equivalent.*

1. *$\underline{N}$ is an $\mathbb{E}$-unitary,*
2. *N_3 is a unitary and $\underline{N}$ is an $\mathbb{E}$-contraction,*
3. *N_3 is a unitary, N_2 is a contraction and $N_1 = N_2^* N_3$.*

Here is a structure theorem for the $\mathbb{E}$-isometries (see Theorem 5.6 and 5.7 in [4]).

Theorem 2.7. *Let $\underline{V} = (V_1, V_2, V_3)$ be a commuting triple of bounded operators. Then the following are equivalent.*

1. *$\underline{V}$ is an $\mathbb{E}$-isometry.*
2. *V_3 is an isometry and $\underline{V}$ is an $\mathbb{E}$-contraction.*
3. *V_3 is an isometry, V_2 is a contraction and $V_1 = V_2^* V_3$.*
4. *(Wold decomposition) $\mathcal{H}$ has a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into reducing subspaces of V_1, V_2, V_3 such that $(V_1|_{\mathcal{H}_1}, V_2|_{\mathcal{H}_1}, V_3|_{\mathcal{H}_1})$ is an $\mathbb{E}$-unitary and $(V_1|_{\mathcal{H}_2}, V_2|_{\mathcal{H}_2}, V_3|_{\mathcal{H}_2})$ is a pure $\mathbb{E}$-isometry.*

3. Canonical decomposition of an $\mathbb{E}$-contraction

Theorem 3.1. *Let (A,B,P) be an $\mathbb{E}$-contraction on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_1$ be the maximal subspace of $\mathcal{H}$ which reduces P and on which P is unitary. Let $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Then $\mathcal{H}_1, \mathcal{H}_2$ reduce A, B; $(A|_{\mathcal{H}_1}, B|_{\mathcal{H}_1}, P|_{\mathcal{H}_1})$ is an $\mathbb{E}$-unitary and $(A|_{\mathcal{H}_2}, B|_{\mathcal{H}_2}, P|_{\mathcal{H}_2})$ is a completely non-unitary $\mathbb{E}$-contraction. The subspaces $\mathcal{H}_1$ or $\mathcal{H}_2$ may equal to the trivial subspace $\{0\}$.*

Proof. It is obvious that if P is a completely non-unitary contraction then $\mathcal{H}_1 = \{0\}$ and if P is a unitary then $\mathcal{H} = \mathcal{H}_1$ and so $\mathcal{H}_2 = \{0\}$. In such cases the theorem is trivial. So let us suppose that P is neither a unitary nor a completely non unitary contraction. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \text{ and } P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, so that P_1 is a unitary and P_2 is completely non-unitary. Since P_2 is completely non-unitary it follows that if $x \in \mathcal{H}$ and

$$\|P_2^n x\| = \|x\| = \|P_2^{*n} x\|, \quad n = 1, 2, \ldots$$

then $x = 0$.

The fact that A and P commute tells us that

$$A_{11}P_1 = P_1A_{11} \qquad A_{12}P_2 = P_1A_{12}\,, \tag{3.1}$$
$$A_{21}P_1 = P_2A_{21} \qquad A_{22}P_2 = P_2A_{22}\,. \tag{3.2}$$

Also by commutativity of B and P we have

$$B_{11}P_1 = P_1B_{11} \qquad B_{12}P_2 = P_1B_{12}\,, \tag{3.3}$$
$$B_{21}P_1 = P_2B_{21} \qquad B_{22}P_2 = P_2B_{22}\,. \tag{3.4}$$

By Lemma 2.5, we have for all $\omega, \beta \in \mathbb{T}$,

$$\rho_1(\omega A, \omega B, \omega^2 P) = (I - P^*P) + (A^*A - B^*B) - 2 \text{ Re } \omega(A - B^*P) \geq 0\,,$$
$$\rho_2(\beta A, \beta B, \beta^2 P) = (I - P^*P) + (B^*B - A^*A) - 2 \text{ Re } \beta(B - A^*P) \geq 0\,.$$

Adding ρ_1 and ρ_2 we get

$$(I - P^*P) - \text{Re } \omega(A - B^*P) - \text{Re } \beta(B - A^*P) \geq 0$$

that is

$$\begin{bmatrix} 0 & 0 \\ 0 & I - P_2^*P_2 \end{bmatrix} - \text{Re } \omega \begin{bmatrix} A_{11} - B_{11}^*P_1 & A_{12} - B_{21}^*P_2 \\ A_{21} - B_{12}^*P_1 & A_{22} - B_{22}^*P_2 \end{bmatrix} - \text{Re } \beta \begin{bmatrix} B_{11} - A_{11}^*P_1 & B_{12} - A_{21}^*P_2 \\ B_{21} - A_{12}^*P_1 & B_{22} - A_{22}^*P_2 \end{bmatrix} \geq 0 \tag{3.5}$$

for all $\omega, \beta \in \mathbb{T}$. Since the matrix in the left hand side of (3.5) is self-adjoint, if we write (3.5) as

$$\begin{bmatrix} R & X \\ X^* & Q \end{bmatrix} \geq 0 \,, \tag{3.6}$$

then

$$\begin{cases} \text{(i) } R\,, Q \geq 0 \text{ and } R = -\text{ Re } \omega(A_{11} - B_{11}^* P_1) - \text{ Re } \beta(B_{11} - A_{11}^* P_1) \\ \text{(ii)} X = -\frac{1}{2}\{\omega(A_{12} - B_{21}^* P_2) + \bar{\omega}(A_{21}^* - P_1^* B_{12}) \\ \qquad\qquad +\beta(B_{12} - A_{21}^* P_2) + \bar{\beta}(B_{21}^* - P_1^* A_{12})\} \\ \text{(iii) } Q = (I - P_2^* P_2) - \text{ Re } \omega(A_{22} - B_{22}^* P_2) - \text{ Re } \beta(B_{22} - A_{22}^* P_2) \,. \end{cases}$$

Since the left hand side of (3.6) is a positive semi-definite matrix for every ω and β, if we choose $\beta = 1$ and $\beta = -1$ respectively then consideration of the $(1,1)$ block reveals that

$$\omega(A_{11} - B_{11}^* P_1) + \bar{\omega}(A_{11}^* - P_1^* B_{11}) \leq 0$$

for all $\omega \in \mathbb{T}$. Choosing $\omega = \pm 1$ we get

$$(A_{11} - B_{11}^* P_1) + (A_{11}^* - P_1^* B_{11}) = 0 \tag{3.7}$$

and choosing $\omega = \pm i$ we get

$$(A_{11} - B_{11}^* P_1) - (A_{11}^* - P_1^* B_{11}) = 0 \,. \tag{3.8}$$

Therefore, from (3.7) and (3.8) we get

$$A_{11} = B_{11}^* P_1 \,,$$

where P_1 is unitary. Similarly, we can show that

$$B_{11} = A_{11}^* P_1 \,.$$

Therefore, $R = 0$. Since (A, B, P) is an $\mathbb{E}$-contraction, $\|B\| \leq 1$ and hence $\|B_{11}\| \leq 1$ also. Therefore, by part-(3) of Theorem 2.6, (A_{11}, B_{11}, P_1) is an $\mathbb{E}$-unitary.

Now we apply Proposition 1.3.2 of [3] to the positive semi-definite matrix in the left hand side of (3.6). This Proposition states that if $R, Q \geq 0$ then $\begin{bmatrix} R & X \\ X^* & Q \end{bmatrix} \geq 0$ if and only if $X = R^{1/2} K Q^{1/2}$ for some contraction K.

Since $R = 0$, we have $X = 0$. Therefore,

$$\omega(A_{12} - B_{21}^* P_2) + \bar{\omega}(A_{21}^* - P_1^* B_{12}) + \beta(B_{12} - A_{21}^* P_2) + \bar{\beta}(B_{21}^* - P_1^* A_{12}) = 0 \,,$$

for all $\omega, \beta \in \mathbb{T}$. Choosing $\beta = \pm 1$ we get

$$\omega(A_{12} - B_{21}^* P_2) + \bar{\omega}(A_{21}^* - P_1^* B_{12}) = 0 \,,$$

for all $\omega \in \mathbb{T}$. With the choices $\omega = 1, i$, this gives

$$A_{12} = B_{21}^* P_2 \,.$$

Therefore, we also have

$$A_{21}^* = P_1^* B_{12} \,.$$

Similarly, we can prove that

$$B_{12} = A_{21}^* P_2 \,, \quad B_{21}^* = P_1^* A_{12} \,.$$

Thus, we have the following equations

$$A_{12} = B_{21}^* P_2 \qquad A_{21}^* = P_1^* B_{12} \tag{3.9}$$

$$B_{12} = A_{21}^* P_2 \qquad B_{21}^* = P_1^* A_{12} \,. \tag{3.10}$$

Thus from (3.9), $A_{21} = B_{12}^* P_1$ and together with the first equation in (3.2), this implies that

$$B_{12}^* P_1^2 = A_{21} P_1 = P_2 A_{21} = P_2 B_{12}^* P_1$$

and hence

$$B_{12}^* P_1 = P_2 B_{12}^* \,. \tag{3.11}$$

From equations in (3.3) and (3.11) we have that

$$B_{12} P_2 = P_1 B_{12} \,, \quad B_{12} P_2^* = P_1^* B_{12}.$$

Thus

$$\begin{aligned} B_{12} P_2 P_2^* &= P_1 B_{12} P_2^* = P_1 P_1^* B_{12} = B_{12} \,, \\ B_{12} P_2^* P_2 &= P_1^* B_{12} P_2 = P_1^* P_1 B_{12} = B_{12} \,, \end{aligned}$$

and so we have

$$P_2 P_2^* B_{12}^* = B_{12}^* = P_2^* P_2 B_{12}^* \,.$$

This shows that P_2 is unitary on the range of B_{12}^* which can never happen because P_2 is completely non-unitary. Therefore, we must have $B_{12}^* = 0$ and so $B_{12} = 0$. Similarly we can prove that $A_{12} = 0$. Also from (3.9), $A_{21} = 0$ and from (3.10), $B_{21} = 0$. Thus with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} , \quad B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} .$$

So, $\mathcal{H}_1$ and $\mathcal{H}_2$ reduce A and B. Also (A_{22}, B_{22}, P_2), being the restriction of the $\mathbb{E}$-contraction (A, B, P) to the reducing subspace $\mathcal{H}_2$, is an $\mathbb{E}$-contraction. Since P_2 is completely non-unitary, (A_{22}, B_{22}, P_2) is a completely non-unitary $\mathbb{E}$-contraction.

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4. Operator model

Wold decomposition breaks an isometry into two parts namely a unitary and a pure isometry (see Section-I, Ch-1, [10]). We have in Theorem 2.7 an analogous decomposition for an $\mathbb{E}$-isometry by which an $\mathbb{E}$-isometry splits into two parts of which one is an $\mathbb{E}$-unitary and the other is a pure $\mathbb{E}$-isometry. The following theorem gives a concrete model for pure $\mathbb{E}$-isometries. Before going to the theorem, we recall the definition of Toeplitz operator with operator-valued kernel.

For a Hilbert space E let $L^2(E)$ be the space of all E-valued square integrable functions on $\mathbb{T}$ and let $H^2(E)$ be the space of analytic elements in $L^2(E)$. Also let $L^\infty(\mathcal{B}(E))$ denote the space of $\mathcal{B}(E)$-valued functions on $\mathbb{T}$ with finite supremum norm. For $\phi \in L^\infty(\mathcal{B}(E))$, the Toeplitz operator T_ϕ with operator-valued symbol ϕ is defined by

$$T_\phi : H^2(E) \to H^2(E)$$
$$T_\phi(f) = P(\phi f)$$

where $f \in H^2(E)$ and P is the projection of $L^2(E)$ onto $H^2(E)$.

Theorem 4.1. *Let $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$ be a pure $\mathbb{E}$-isometry acting on a Hilbert space $\mathcal{H}$ and let A_1, A_2 denote the fundamental operators of the adjoint $(\hat{T_1}^*, \hat{T_2}^*, \hat{T_3}^*)$. Then there exists a unitary $U : \mathcal{H} \to H^2(\mathcal{D}_{\hat{T_3}^*})$ such that*

$$\hat{T}_1 = U^* T_\varphi U, \quad \hat{T}_2 = U^* T_\psi U \text{ and } \hat{T}_3 = U^* T_z U,$$

where $\varphi(z) = G_1^* + G_2 z$, $\psi(z) = G_2^* + G_1 z$, $z \in \mathbb{T}$ and G_1, G_2 are restrictions of UA_1U^* and UA_2U^* to the defect space $\mathcal{D}_{\hat{T}_3^*}$. Moreover, A_1, A_2 satisfy

(1) $[A_1, A_2] = 0$;
(2) $[A_1^*, A_1] = [A_2^*, A_2]$; *and*
(3) $\|A_1^* + A_2 z\| \leq 1$ *for all* $z \in \mathbb{D}$.

Conversely, if A_1 and A_2 are two bounded operators on a Hilbert space E satisfying the above three conditions, then $(T_{A_1^*+A_2z}, T_{A_2^*+A_1z}, T_z)$ on $H^2(E)$ is a pure $\mathbb{E}$-isometry.

See Theorem 3.3 in [8] for a proof to this theorem. The following dilation theorem was proved in [4] and for a proof one can see Theorem 6.1 in [4].

Theorem 4.2. *Let (A, B, P) be a tetrablock contraction on $\mathcal{H}$ with fundamental operators F_1 and F_2. Let $\mathcal{D}_P$ be the closure of the range of D_P. Let $\mathcal{K} = \mathcal{H} \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \cdots = \mathcal{H} \oplus l^2(\mathcal{D}_P)$. Consider the operators V_1, V_2 and V_3 defined on $\mathcal{K}$ by*

$$V_1(h_0, h_1, h_2, \dots) = (Ah_0, F_2^* D_P h_0 + F_1 h_1, F_2^* h_1 + F_1 h_2, F_2^* h_2 + F_1 h_3, \dots)$$
$$V_2(h_0, h_1, h_2, \dots) = (Bh_0, F_1^* D_P h_0 + F_2 h_1, F_1^* h_1 + F_2 h_2, F_1^* h_2 + F_2 h_3, \dots)$$
$$V_3(h_0, h_1, h_2, \dots) = (Ph_0, D_P h_0, h_1, h_2, \dots).$$

Then

(1) $\underline{V} = (V_1, V_2, V_3)$ *is a minimal tetrablock isometric dilation of* (A, B, P) *if* $[F_1, F_2] = 0$ *and* $[F_1, F_1^*] = [F_2, F_2^*]$.
(2) *If there is a tetrablock isometric dilation* $\underline{W} = (W_1, W_2, W_3)$ *of* (A, B, P) *such that* W_3 *is the minimal isometric dilation of* P*, then* $\underline{W}$ *is unitarily equivalent to* $\underline{V}$*. Moreover,* $[F_1, F_2] = 0$ *and* $[F_1, F_1^*] = [F_2, F_2^*]$.

The following result of one variable dilation theory is necessary for the proof of the model theorem for $\mathbb{E}$-contractions and since the result is well-known we do not give a proof here.

Proposition 4.3. *If P is a contraction and W is its minimal isometric dilation then P^* and W^* have defect spaces of same dimension.*

The next theorem is the main result of this section and it provides a model for the $\mathbb{E}$-contractions which satisfy some conditions.

Theorem 4.4. *Let (A,B,P) be an $\mathbb{E}$-contraction on a Hilbert space $\mathcal{H}$ and let F_1, F_2 and F_{1*}, F_{2*} be respectively the fundamental operators of (A,B,P) and (A^*,B^*,P^*). Let F_{1*}, F_{2*} satisfy $[F_{1*},F_{2*}]=0$ and $[F_{1*}^*,F_{1*}]=[F_{2*}^*,F_{2*}]$. Let (T_1,T_2,T_3) on $\mathcal{K}_* = \mathcal{H}\oplus\mathcal{D}_{P^*}\oplus\mathcal{D}_{P^*}\oplus\cdots$ be defined as*

$$T_1 = \begin{bmatrix} A & D_{P^*}F_{2*} & 0 & 0 & \cdots \\ 0 & F_{1*}^* & F_{2*} & 0 & \cdots \\ 0 & 0 & F_{1*}^* & F_{2*} & \cdots \\ 0 & 0 & 0 & F_{1*}^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad T_2 = \begin{bmatrix} B & D_{P^*}F_{1*} & 0 & 0 & \cdots \\ 0 & F_{2*}^* & F_{1*} & 0 & \cdots \\ 0 & 0 & F_{2*}^* & F_{1*} & \cdots \\ 0 & 0 & 0 & F_{2*}^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$T_3 = \begin{bmatrix} P & D_{P^*} & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ 0 & 0 & 0 & I & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

(1) *(T_1,T_2,T_3) is an $\mathbb{E}$-co-isometry, $\mathcal{H}$ is a common invariant subspace of T_1,T_2,T_3 and $T_1|_{\mathcal{H}}=A$, $T_2|_{\mathcal{H}}=B$ and $T_3|_{\mathcal{H}}=P$;*
(2) *there is an orthogonal decomposition $\mathcal{K}_*=\mathcal{K}_1\oplus\mathcal{K}_2$ into reducing subspaces of T_1, T_2 and T_3 such that $(T_1|_{\mathcal{K}_1},T_2|_{\mathcal{K}_1},T_3|_{\mathcal{K}_1})$ is an $\mathbb{E}$-unitary and $(T_1|_{\mathcal{K}_2},T_2|_{\mathcal{K}_2},T_3|_{\mathcal{K}_2})$ is a pure $\mathbb{E}$-co-isometry;*
(3) *$\mathcal{K}_2$ can be identified with $H^2(\mathcal{D}_{T_3})$, where $\mathcal{D}_{T_3}$ has same dimension as that of $\mathcal{D}_P$. The operators $T_1|_{\mathcal{K}_2}$, $T_2|_{\mathcal{K}_2}$ and $T_3|_{\mathcal{K}_2}$ are respectively unitarily equivalent to $T_{G_1+G_2^*\bar{z}}$, $T_{G_2+G_1^*\bar{z}}$ and $T_{\bar{z}}$ defined on $H^2(\mathcal{D}_{T_3})$, G_1,G_2 being the fundamental operators of (T_1,T_2,T_3).*

Proof. We apply Theorem 4.2 to (A^*,B^*,P^*) to obtain a minimal $\mathbb{E}$-isometric dilation for (A^*,B^*,P^*). If we denote this $\mathbb{E}$-isometric dilation by (V_{1*},V_{2*},V_{3*}) then it is evident from Theorem 4.2 that each V_{i*} is defined on $\mathcal{K}_* = \mathcal{H}\oplus\mathcal{D}_{P^*}\oplus\mathcal{D}_{P^*}\oplus\cdots$ and with respect to this decomposition

$$V_{1*}=\begin{bmatrix} A^* & 0 & 0 & 0 & \dots \\ F_{2*}^*D_{P^*} & F_{1*} & 0 & 0 & \dots \\ 0 & F_{2*}^* & F_{1*} & 0 & \dots \\ 0 & 0 & F_{2*}^* & F_{1*} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}, V_{2*}=\begin{bmatrix} B^* & 0 & 0 & 0 & \dots \\ F_{1*}^*D_{P^*} & F_{2*} & 0 & 0 & \dots \\ 0 & F_{1*}^* & F_{2*} & 0 & \dots \\ 0 & 0 & F_{1*}^* & F_{2*} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix},$$

$$V_{3*}=\begin{bmatrix} P^* & 0 & 0 & 0 & \dots \\ D_{P^*} & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Obviously $(T_1^*, T_2^*, T_3^*) = (V_{1*}, V_{2*}, V_{3*})$. It is clear from the block matrices of T_i that $\mathcal{H}$ is a common invariant subspace of each T_i and $T_1|_{\mathcal{H}} = A$, $T_2|_{\mathcal{H}} = B$ and $T_3|_{\mathcal{H}} = P$. Again since (T_1^*, T_2^*, T_3^*) is an $\mathbb{E}$-isometry, by Theorem 2.7, there is an orthogonal decomposition $\mathcal{K}_* = \mathcal{K}_1 \oplus \mathcal{K}_2$ into reducing subspaces of T_i such that $(T_1|_{\mathcal{K}_1}, T_2|_{\mathcal{K}_1}, T_3|_{\mathcal{K}_1})$ is an $\mathbb{E}$-unitary and $(T_1|_{\mathcal{K}_2}, T_2|_{\mathcal{K}_2}, T_3|_{\mathcal{K}_2})$ is a pure $\mathbb{E}$-co-isometry.

If we denote $(T_1|_{\mathcal{K}_1}, T_2|_{\mathcal{K}_1}, T_3|_{\mathcal{K}_1})$ by (T_{11}, T_{12}, T_{13}) and $(T_1|_{\mathcal{K}_2}, T_2|_{\mathcal{K}_2}, T_3|_{\mathcal{K}_2})$ by (T_{21}, T_{22}, T_{23}), then with respect to the orthogonal decomposition $\mathcal{K}_* = \mathcal{K}_1 \oplus \mathcal{K}_2$ we have that

$$T_1=\begin{bmatrix} T_{11} & 0 \\ 0 & T_{21} \end{bmatrix}, T_2=\begin{bmatrix} T_{12} & 0 \\ 0 & T_{22} \end{bmatrix}, T_3=\begin{bmatrix} T_{13} & 0 \\ 0 & T_{23} \end{bmatrix}.$$

The fundamental equations $T_1 - T_2^*T_3 = D_{T_3}X_1D_{T_3}$ and $T_2 - T_1^*T_3 = D_{T_3}X_2D_{T_3}$ clearly become

$$\begin{bmatrix} T_{11}-T_{12}^*T_{13} & 0 \\ 0 & T_{21}-T_{22}^*T_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D_{T_{23}}X_{12}D_{T_{23}} \end{bmatrix},\ X_1=\begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix}$$

and

$$\begin{bmatrix} T_{12}-T_{11}^*T_{13} & 0 \\ 0 & T_{22}-T_{21}^*T_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D_{T_{23}}X_{22}D_{T_{23}} \end{bmatrix},\ X_2=\begin{bmatrix} X_{21} \\ X_{22} \end{bmatrix}.$$

Thus T_3 and T_{23} have same defect spaces, that is $\mathcal{D}_{T_3}$ and $\mathcal{D}_{T_{23}}$ are same and consequently (T_1, T_2, T_3) and (T_{21}, T_{22}, T_{23}) have the same fundamental operators. Now we apply Theorem 4.1 to the pure $\mathbb{E}$-isometry $(T_{21}^*, T_{22}^*, T_{23}^*) = (T_1^*|_{\mathcal{K}_2}, T_2^*|_{\mathcal{K}_2}, T_3^*|_{\mathcal{K}_2})$ and get the following:

(i) $\mathcal{K}_2$ can be identified with $H^2(\mathcal{D}_{T_{23}})(= H^2(\mathcal{D}_{T_3}))$;

(ii) $(T_{21}^*, T_{22}^*, T_{23}^*)$ can be identified with the commuting triple of Toeplitz operators $(T_{G_1^*+G_2z}, T_{G_2^*+G_1z}, T_z)$ defined on $H^2(\mathcal{D}_{T_3})$, where G_1, G_2 are the fundamental operators of (T_1, T_2, T_3).

Therefore, $T_1|_{\mathcal{K}_2}$, $T_2|_{\mathcal{K}_2}$ and $T_3|_{\mathcal{K}_2}$ are respectively unitarily equivalent to $T_{G_1+G_2^*\bar{z}}$, $T_{G_2+G_1^*\bar{z}}$ and $T_{\bar{z}}$ defined on $H^2(\mathcal{D}_{T_3})$. The fact that $\mathcal{D}_{T_3}$ and $\mathcal{D}_P$ have same dimensions follows from Proposition 4.3 as T_3^* is the minimal isometric dilation of P^*. ■

Remark 4.5. Theorem 4.4 is obtained by applying Theorem 4.1 and Theorem 4.2 (which is Theorem 6.1 in [4]). Theorem 4.1 has intersection with Theorem 5.10 in [4]. Theorem 5.10 in [4] gives the form of a pure $\mathbb{E}$-isometry stated in Theorem 4.1. In Theorem 4.1 it has been shown that the operator-valued kernels τ_1, τ_2 associated with the Topelitz operators occurring in Theorem 5.10 of [4] can be identified with the fundamental operators of the adjoint of the mentioned pure $\mathbb{E}$-isometry.

Acknowledgement. The author greatly appreciates the warm and generous hospitality provided by Indian Statistical Institute, Delhi during the course of the work.

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Department of Mathematics, Indian Institute of Technology Bombay, Mumbai - 400076, India.
E-mail address: sourav@math.iitb.ac.in, souravmaths@gmail.com